Quantifying Transversality by Measuring the Robustness of Intersections *

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Abstract

By definition, transverse intersections are stable under infinitesimal perturbations. Using persistent homology, we extend this notion to sizeable perturbations. Specifically, we assign to each homology class of the intersection its robustness, the magnitude of a perturbation necessary to kill it, and prove that robustness is stable. Among the applications of this result is a stable notion of robustness for fixed points of continuous mappings and a statement of stability for contours of smooth mappings.

Keywords. Smooth mappings, transversality, fixed points, contours, homology, filtrations, zigzag modules, persistence, stability.

1 Introduction

The main new concept in this paper is a quantification of the classically differential notion of transversality. This is achieved by extending persistence from filtrations of homology groups to zigzag modules of well groups.

Motivation. In hind-sight, we place the starting point for the work described in this paper at the difference between qualitative and quantitative statements and their relevance in the sciences; see eg. the discussion in Thom's book [15, Chapters 1.3 and 13.8]. It appears the conscious mind thinks in qualitative terms, delegating the quantitative details to the unconscious, if possible. In the sciences, quantitative statements are a requirement for testing a hypothesis. Without such a test, the hypothesis is not falsifiable and, by popular philosophical interpretation, not scientific [13]. The particular field discussed in [15] is the mathematical study of singularities of smooth mappings, which is dominated by qualitative statements. We refer to the seminal papers by Whitney [17, 18] and the book by Arnold [1] for introductions. A unifying concept in this field is the transversality of an intersection between two spaces. Its roots go far back in history and appear among others in the work of Poincaré about a century ago. It took a good development toward its present form under Pontryagin and Whitney; see eg. [14]. In this review of Zeeman's book [19], Smale criticizes the unscientific aspects of the work promoted in then popular area of catastrophe theory, thus significantly contributing to the discussion of qualitative versus quantitative statements and to the fate of that field. At the same time, Smale points to positive aspects and stresses the importance of the concept of transversality in the study of singularities. In a nutshell, an intersection is transverse if it forms a non-zero angle and is therefore stable under infinitesimal perturbations; see Section 2 for a formal definition.

Results. We view our work as a measure theoretic extension of this essentially differential concept. We extend by relaxing the requirements on the perturbations to continuous but not necessarily smooth mappings. At the same time, we are more tolerant to changes in the intersection. To rationalize this tolerance, we measure intersections using real numbers as opposed to 0 for non-existence and 1 for existence. The measurements are made using the concept of persistent homology; see [7] for the original paper and [5] for a recent survey. However, we have need for modifications and use the extension of persistence from filtrations to zigzag modules as proposed in [2]. An important property of persistence, as originally defined for filtrations, is the stability of its diagrams; see [4] for the original proof. There is no comparably general result known for zigzag modules. Our main result is a step in this direction. Specifically, we view the following

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as the main contributions of this paper:

1. the introduction of the zigzag module of well groups and the proof that their diagrams are stable;
2. the interpretation of the values in these diagrams as measurements of transversality, referred to as the robustness of intersections;
3. the application of these results to fixed points and periodic orbits of continuous mappings.

Our results have also ramifications in the study of the set of critical values, the apparent contour of a smooth mapping. Specifically, the stability of the diagrams mentioned above completes the proof of the stability of the apparent contour of a smooth mapping from an orientable 2-manifold to the plane given in [8]. The need for this stability result was indeed what triggered the development described in this paper.

Outline. Section 2 provides the relevant background. Section 3 explains how we measure robustness using well groups and zigzag modules. Section 4 proves our main result, the stability of the diagrams defined by the modules. Section 5 discusses applications. Section 6 concludes the paper.

2 Background

We need the algebraic concept of persistent homology to extend the differential notion of transversality as explained in the introduction. In this section, we give a formal definition of transversality, referring to [11] for general background in differential topology. We also introduce homology and persistent homology, referring to [12] for general background in classic algebraic topology and to [6] for a text on computational topology.

Transversality. Let $\mathbb{X}, \mathbb{Y}$ be manifolds, $f : \mathbb{X} \to \mathbb{Y}$ a smooth mapping, and $\mathbb{A} \subseteq \mathbb{Y}$ a smoothly embedded submanifold of the range. We assume the manifolds have finite dimension and no boundary, writing $m = \dim \mathbb{X}$, $n = \dim \mathbb{Y}$, and $k = \dim \mathbb{A}$. Given a point $x \in \mathbb{X}$ and a smooth curve $\gamma : \mathbb{R} \to \mathbb{X}$ with $\gamma(0) = x$, we call $\dot\gamma(0)$ the *tangent vector* of γ at x. Varying the curve, we get a set of tangent vectors called the *tangent space* of $\mathbb{X}$ at x, denoted as $\mathrm{T}_x\mathbb{X}$. Composing the curves with the mapping, $f \circ \gamma : \mathbb{R} \to \mathbb{Y}$, we get a subset of all smooth curves passing through $y = f(x) = f \circ \gamma(0)$. The *derivative* of f at x is $\mathrm{D}f(x) : \mathrm{T}_x\mathbb{X} \to \mathrm{T}_y\mathbb{Y}$ defined by mapping the tangent vector of γ at x to the tangent vector of $f \circ \gamma$ at y. The derivative is a linear map and its image is a subspace of $\mathrm{T}_y\mathbb{Y}$. The dimensions of the tangent spaces are $m = \dim \mathrm{T}_x\mathbb{X}$ and $n = \dim \mathrm{T}_y\mathbb{Y}$, which implies that the dimension of the image of the derivative is $\dim \mathrm{D}f(x)(\mathrm{T}_x\mathbb{X}) \leq \min\{m, n\}$.

We are interested in properties of f that are stable under perturbations. We call a property *infinitesimally stable* if for every smooth homotopy, $F : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ with $f_0 = f$, there is a real number $\delta > 0$ such that f_t possesses the same property for all $t < \delta$, where $f_t(x) = F(x, t)$ for all $x \in \mathbb{X}$. An important example of such a property is the following. The mapping f is *transverse* to $\mathbb{A}$, denoted as $f \pitchfork \mathbb{A}$, if for each $x \in \mathbb{X}$ with $f(x) \in \mathbb{A}$, the image of the derivative of f at x together with the tangent space of $\mathbb{A}$ at $a = f(x)$ spans the tangent space of $\mathbb{Y}$ at a. More formally, $f \pitchfork \mathbb{A}$ if $\mathrm{D}f(x)(\mathrm{T}_x\mathbb{X}) + \mathrm{T}_a\mathbb{A} = \mathrm{T}_a\mathbb{Y}$. It is plausible but also true that transversality is an infinitesimally stable property.

Product spaces. It is convenient to recast transversality in terms of intersections of subspaces of the product space $\mathbb{X} \times \mathbb{Y}$, a manifold of dimension $m + n$. Consider the graphs of f and of its restriction to $\mathbb{A}$,

$$\begin{aligned} \mathrm{gf}\, f &= \{(x, y) \in \mathbb{X} \times \mathbb{Y} \mid y = f(x)\}; \\ \mathrm{gf}\, f|_{\mathbb{A}} &= \{(x, a) \in \mathbb{X} \times \mathbb{A} \mid a = f(x)\}. \end{aligned}$$

The intersection of interest is between $\mathrm{gf}\, f$ and $\mathbb{X} \times \mathbb{A}$, two manifolds of dimensions m and $m + k$ embedded in $\mathbb{X} \times \mathbb{Y}$. This intersection is the graph of $f|_{\mathbb{A}}$, which is homeomorphic to the preimage of $\mathbb{A}$. See Figure 1 for an example in which

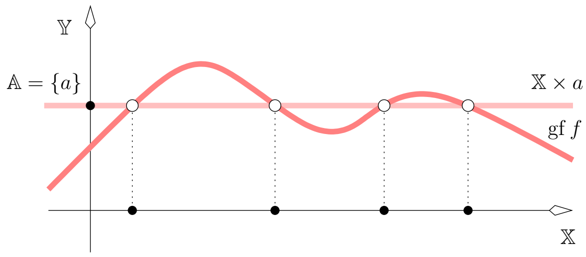

Figure 1: The preimage of a, consisting of four points on the horizontal axis representing $\mathbb{X}$, is homeomorphic to the intersection of the curve with the horizontal line passing through a.

$m = n = 1$ and $k = 0$. Here, $\mathrm{T}_a\mathbb{A} = 0$ and transversality requires that whenever the curve, $\mathrm{gf}\, f$, intersects the line, $\mathbb{X} \times \mathbb{A}$, it crosses at a non-zero angle. This is the case in Figure 1 which implies that having a cardinality four preimage of a is an infinitesimally stable property of f. Nevertheless, the left two intersection points are clearly more stable than the right two intersection points, but we will need some algebra to give precise meaning to this statement.

Homology. The algebraic language of homology is a means to define and count holes in a topological space. We think of it as a functor that maps a space to an abelian group and a continuous map between spaces to a homomorphism between the corresponding groups. We have such a functor for each dimension, p. It is convenient to combine the homology groups of all dimensions into a single algebraic structure. Writing $\mathsf{H}_p(\mathbb{X})$ for the p-dimensional homology group of the topological space $\mathbb{X}$, we form a graded group by taking direct sums,

$$\mathsf{H}(\mathbb{X}) = \bigoplus_{p \geq 0} \mathsf{H}_p(\mathbb{X}).$$

To simplify language and notation, we will suppress dimensions and refer to $\mathsf{H}(\mathbb{X})$ is the *homology group* of $\mathbb{X}$. Its elements are formally written as polynomials, $\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots$, where α_p is an p-dimensional homology class and only finitely many of the classes are non-zero. As usual, adding two polynomials is done componentwise. The groups $\mathsf{H}_p(\mathbb{X})$ depend on a choice of coefficient group. The theory of persistence introduced below requires we use field coefficients. An example is modulo two arithmetic in which the field is $\mathbb{Z}_2 = \{0, 1\}$. The p-dimensional group is then a vector space, $\mathsf{H}_p(\mathbb{X}) \simeq \mathbb{Z}_2^{\beta_p}$, and its rank, the dimension of the vector space, is the *p-th Betti number*, $\beta_p = \beta_p(\mathbb{X})$. Similarly, $\mathsf{H}(\mathbb{X})$ is a vector space of dimension $\sum_{p \geq 0} \beta_p$. We say $\mathbb{X}$ and $\mathbb{Y}$ *have the same homology* if there is an isomorphism between $\mathsf{H}(\mathbb{X})$ and $\mathsf{H}(\mathbb{Y})$ whose restrictions to the components are isomorphisms. Equivalently, $\beta_p(\mathbb{X}) = \beta_p(\mathbb{Y})$ for all non-negative integers p.

Persistent homology. Now suppose we have a finite sequence of nested spaces, $\mathbb{X}_1 \subseteq \mathbb{X}_2 \subseteq \ldots \subseteq \mathbb{X}_\ell$. Writing $\Phi_i = \mathsf{H}(\mathbb{X}_i)$ for the homology group of the i-th space, we get a sequence of vector spaces connected from left to right by homomorphic maps induced by inclusion:

$$\Phi : \Phi_1 \to \Phi_2 \to \ldots \to \Phi_\ell.$$

We call this sequence a *filtration*. To study the evolution of the homology classes as we progress from left to right in the filtration, we let $\varphi_{i,j}$ be the composition of the maps between Φ_i and Φ_j, for $i \leq j$. We say a class $\alpha \in \Phi_i$ is *born* at Φ_i if it does not belong to the image of $\varphi_{i-1,i}$. Furthermore, this class α *dies entering* Φ_j if $\varphi_{i,j-1}(\alpha)$ does not belong to the image of $\varphi_{i-1,j-1}$ but $\varphi_{i,j}(\alpha)$ does belong to the image of $\varphi_{i-1,j}$. We call the images of the maps $\varphi_{i,j}$ the *persistent homology groups* of the filtration and record the evolution of the homology classes in the *persistence diagram* of the filtration, denoted as $\mathrm{Dgm}(\Phi)$. This is a multiset of points in the extended plane, $\bar{\mathbb{R}}^2 = (\mathbb{R} \cup \{-\infty, \infty\})^2$. Marking an increase in rank on the horizontal, birth axis and a drop in rank on the vertical, death axis, each point represents the birth and the death of a generator and records where these events happen; see Figure 2. For technical reasons that will become clear shortly, we add infinitely many copies of each point on the diagonal to the diagram. Given an index, i, we can read off the rank of $\mathsf{H}(\mathbb{X}_i)$ by counting the points in the half-open upper-left quadrant, $[-\infty, i] \times (i, \infty]$, anchored at the point (i, i) on the diagonal. More generally, the rank of the image of $\varphi_{i,j}$ equals the number of points in the upper-left quadrant anchored at (i, j).

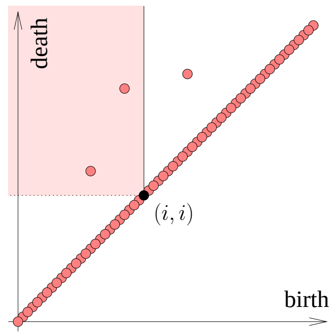

Figure 2: The three off-diagonal points represent the birth and death of three generators. The number of points in the upper-left quadrant equals the rank of the corresponding homology group.

Stability. Consider now the case in which the spaces in the sequence are sublevel sets of a real valued function $\varphi : \mathbb{X} \to \mathbb{R}$, that is, there are values r_i such that $\mathbb{X}_i = \varphi^{-1}(\infty, r_i]$ for each i. A *homological critical value* of φ is a value r such that for every sufficiently small $\delta > 0$, the homomorphism from $\mathsf{H}(\varphi^{-1}(-\infty, r - \delta])$ to $\mathsf{H}(\varphi^{-1}(-\infty, r + \delta])$ induced by inclusion is not an isomorphism. We suppose φ is *tame* by which we mean that each sublevel set has finite rank homology and there are only finitely many homological critical values, denoted as $r_1 < r_2 < \ldots < r_\ell$. We can represent the evolution of the homology classes by the finite filtration consisting of the groups $\Phi_i = \mathsf{H}(\varphi^{-1}(-\infty, r_i])$ for $1 \leq i \leq \ell$ and by the persistence diagram of that filtration, $D = \mathrm{Dgm}(\Phi)$. Letting $\psi : \mathbb{X} \to \mathbb{R}$ be another tame function, we get another filtraton, Ψ, and another persistence diagram, $E = \mathrm{Dgm}(\Psi)$. The *bottleneck distance* between the two is the infimum, over all bijections, $\mu : D \to E$, of the L_∞-length of the longest edge in the matching,

$$W_\infty(D, E) = \inf_{\mu} \sup_{a \in D} \|a - \mu(a)\|_\infty.$$

An important result is the stability of the persistence diagram under perturbations of the function.

STABILITY THEOREM FOR TAME FUNCTIONS [4]. Let φ and ψ be tame, real-valued functions on $\mathbb{X}$. Then the bottleneck distance between their persistence diagrams is bounded from above by $\|\varphi - \psi\|_\infty$.

Here, $\|\varphi - \psi\|_\infty = \sup_{x \in \mathbb{X}} |\varphi(x) - \psi(x)|$, as usual. The original form of this result is slightly stronger as it restricts itself to dimension preserving bijections. The theorem implies that the bottleneck distance between the diagrams defined by φ and ψ goes to zero as the difference between the two functions approaches zero.

3 Measuring Robustness

The main new concept in this section is the well diagram of the distance function defined by a mapping $f : \mathbb{X} \to \mathbb{Y}$ and a submanifold $\mathbb{A} \subseteq \mathbb{Y}$. It encodes the persistent homology of the preimage of the submanifold. We begin by setting the stage and introducing the well group of a sublevel set.

Admissible mappings. Assume $\mathbb{Y}$ is a Riemannian manifold and write $\|y - a\|_{\mathbb{Y}}$ for the distance between the points $y, a \in \mathbb{Y}$ assigned by the associated metric. Let $f_{\mathbb{A}} : \mathbb{X} \to \mathbb{R}$ be defined by mapping each point x to the distance of its image from $\mathbb{A}$, that is,

$$f_{\mathbb{A}}(x) \quad = \quad \inf_{a \in \mathbb{A}} \|f(x) - a\|_{\mathbb{Y}}.$$

We call $f_{\mathbb{A}}$ the *distance function* defined by f and $\mathbb{A}$. The *level set* of $f_{\mathbb{A}}$ at a value r is the preimage of that value, $f_{\mathbb{A}}^{-1}(r)$. The *sublevel set* for the same value, r, is the union of level sets at values at most r or, equivalently, the preimage of $[0, r]$. Writing $\mathbb{A}^r$ for the set of points at distance r or less from $\mathbb{A}$, we have $f_{\mathbb{A}}^{-1}[0, r] = f^{-1}(\mathbb{A}^r)$.

In this paper, we limit the class of mappings to those with manageable properties. While our goal is a statement of our results in a context that is sufficiently broad to support interesting applications, we are aware of the technical burden that comes with generality. We hope that the following class of mappings gives a happy median between the conflicting goals of generality and transparency.

DEFINITION. Let $\mathbb{X}$ be an m-manifold, $\mathbb{Y}$ a Riemannian n-manifold, and $\mathbb{A}$ a k-dimensional submanifold of $\mathbb{Y}$. A continuous mapping $f : \mathbb{X} \to \mathbb{Y}$ is *admissible* if $f^{-1}(\mathbb{A})$ has a finite rank homology group.

Requiring that the preimage of $\mathbb{A}$ has finite rank homology is strictly weaker than demanding that the distance function defined by f and $\mathbb{A}$ is tame.

Well groups. Let $h : \mathbb{X} \to \mathbb{Y}$ be a mapping homotopic to f, that is, there is a continuous mapping $H : \mathbb{X} \times [0, 1] \to \mathbb{Y}$ with $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in \mathbb{X}$. We call h a *ρ-perturbation* of f if $\|h - f\|_\infty \leq \rho$, where the norm of the difference is the supremum over all $x \in \mathbb{X}$ of the distance between $h(x)$ and $f(x)$ in $\mathbb{Y}$. The preimage of $\mathbb{A}$ under a ρ-perturbation is contained in the preimage of $\mathbb{A}^\rho$ under f. Writing this in terms of distance functions, we have $h_{\mathbb{A}}^{-1}(0) \subseteq f_{\mathbb{A}}^{-1}[0, \rho]$. This inclusion induces a homomorphism between the corresponding homology groups,

$$\mathsf{j}_h : \mathsf{H}(h_{\mathbb{A}}^{-1}(0)) \to \mathsf{F}(\rho),$$

where we simplify notation by writing $\mathsf{F}(\rho)$ for $\mathsf{H}(f_{\mathbb{A}}^{-1}[0, \rho])$. The image of this map, denoted as $\mathrm{im}\, \mathsf{j}_h$, is a subgroup of $\mathsf{F}(\rho)$. The intersection of subgroups is again a subgroup, which motivates the following definition.

DEFINITION. The *well group* of $f_{\mathbb{A}}^{-1}[0, r]$ is the largest subgroup $\mathsf{U}(r) \subseteq \mathsf{F}(r)$ such that the image of $\mathsf{U}(r)$ in $\mathsf{F}(\rho)$ is contained in $\bigcap_{h:\mathbb{X}\to\mathbb{Y}} \mathrm{im}\, \mathsf{j}_h$, where h ranges over all ρ-perturbations of f and $\rho = r + \delta$ for a sufficiently small $\delta > 0$.

The reason for using ρ- instead of r-perturbations is technical and will become clear later. The requirement that the perturbations be homotopic to f is not used in the proofs and can therefore be dropped. However, removing the requirement changes the well groups and therefore the meaning of our results. Similarly, we may obtain additional variants of our results by modifying the definition of a ρ-perturbation in other ways.

Example. To illustrate the definitions, let us consider again the example in Figure 1. The preimage of $\mathbb{A} = \{a\}$ is a set of four points. The distance function, $f_a : \mathbb{X} \to \mathbb{R}$, has three homological critical values, $r_1 > r_2 > r_3$, with r_i the value of the critical point of f between the i-th and $(i+1)$-st points of $f_a^{-1}(0)$ from the left. Table 1 shows the ranks of $\mathsf{F}(r)$ and $\mathsf{U}(r)$ for values of r in the four intervals delimited by the homological critical values. When r passes from smaller to greater than r_3, two intervals of the sublevel set merge into one. We thus go from four to three intervals; see the first two numbers in the first row of Table 1. At the same time, the rank of the well group drops from four to two. Similar differences between $\mathsf{F}(r)$ and $\mathsf{U}(r)$ can be observed when r passes r_2 and finally r_1.

	$[0, r_3)$	$[r_3, r_2)$	$[r_2, r_1)$	$[r_1, \infty)$
$\mathsf{F}(r)$	4	3	2	1
$\mathsf{U}(r)$	4	2	2	0

Table 1: The ranks of the homology and well groups defined for the mapping f and the submanifold $\mathbb{A} = \{a\}$ in Figure 1.

Terminal critical values. Recall that we assume the mapping $f : \mathbb{X} \to \mathbb{Y}$ is admissible. The initial homology group, $\mathsf{F}(0) = \mathsf{H}(f_{\mathbb{A}}^{-1}(0))$, has therefore finite rank, and because $\mathsf{U}(0) \subseteq \mathsf{F}(0)$, the initial well group has finite rank. Imagine we grow the sublevel set by gradually increasing r from zero to infinity. Since the admissibility of f does not imply the tameness of the distance function, this leaves open the possibility that $f_{\mathbb{A}}$ has infinitely many homological critical values. We call a radius, r, a *terminal critical value* of $f_{\mathbb{A}}$ if for every sufficiently small $\delta > 0$ the homomorphism from $\mathsf{F}(r-\delta)$ to $\mathsf{F}(r+\delta)$ applied to $\mathsf{U}(r-\delta)$ does not give $\mathsf{U}(r+\delta)$. In contrast to the homological critical values, there can only be a finite number of terminal critical values. To see this, we note that the set of images whose common intersection is the well group cannot decrease and the rank of the well group can therefore not increase. To state this relationship between well groups more formally, we write $\mathsf{f}(r,s) : \mathsf{F}(r) \to \mathsf{F}(s)$ for the homomorphism induced by inclusion.

SHRINKING WELLNESS LEMMA. For each choice of radii $0 \leq r \leq s$, the image of the well group at r contains the well group at s, that is, $\mathsf{U}(s) \subseteq \mathsf{f}(r,s)(\mathsf{U}(r))$.

It follows the only way the well group can change is by lowering its rank. Since we start with a finite rank well group at $r = 0$, there can only be finitely many terminal critical values, which we denote as $u_1 < u_2 < \ldots < u_l$. To this sequence, we add $u_0 = 0$ on the left and $u_{l+1} = \infty$ on the right. It is convenient to index the homology groups and the well groups accordingly, writing $\mathsf{F}_i = \mathsf{F}(r_i)$ and $\mathsf{U}_i = \mathsf{U}(r_i)$ for all i. To these sequence, we add $\mathsf{F}_{-1} = \mathsf{U}_{-1} = 0$ on the left and $\mathsf{F}_{l+2} = \mathsf{U}_{l+2} = 0$ on the right. Furthermore, we write $\mathsf{f}_{i,j} : \mathsf{F}_i \to \mathsf{F}_j$ for all feasible choices of $i \leq j$.

Well module. In contrast to the homology groups, the well groups of the sublevel sets do not form a filtration. Instead, they form a special kind of zigzag module. By definition of terminal critical values, the rank of U_i exceeds the rank of U_{i+1}. The rank of the image, $\mathsf{f}_{i,i+1}(\mathsf{U}_i)$, is somewhere between these two ranks. We call a difference between U_i and its image a *conventional death*, in which a class maps to zero, and a difference between the image and U_{i+1} an *unconventional death*, in which the image of a class lies outside the next well group. We capture both cases by inserting a new group between the contiguous well groups; see Figure 3. To

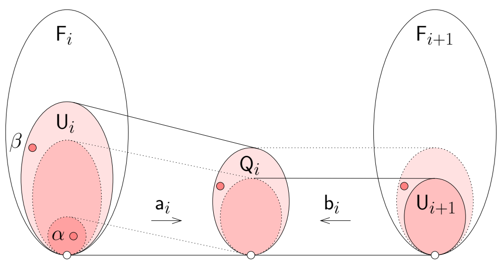

Figure 3: Connecting two consecutive well groups to the quotient group introduced between them. The class α dies a conventional death and the class β dies an unconventional death.

this end, we consider the restriction of $f_{i,i+1}$ to U_i and in particular its kernel, $\mathsf{K}_i = \mathsf{U}_i \cap \ker f_{i,i+1}$, which we refer to as the *vanishing subgroup* of U_i. Using this subgroup, we construct $\mathsf{Q}_i = \mathsf{U}_i / \mathsf{K}_i$. The forward map, $\mathsf{a}_i : \mathsf{U}_i \to \mathsf{Q}_i$, is defined by mapping a class ξ to $\xi + \mathsf{K}_i$. It is clearly surjective. The backward map, $\mathsf{b}_i : \mathsf{U}_{i+1} \to \mathsf{Q}_i$, is defined by mapping a class η to $\xi + \mathsf{K}_i$, where ξ belongs to $\mathsf{f}_{i,i+1}^{-1}(\eta)$. This map is clearly injective. Instead of a filtration in which all maps go from left to right, we get a sequence in which the maps alternate between going forward and backward. As indicated below, every other group in the sequence is a subgroup of the corresponding homology group,

$$\begin{array}{ccccccccc} \mathsf{Q}_{i-1} & \stackrel{\mathsf{b}_{i-1}}{\longleftarrow} & \mathsf{U}_i & \stackrel{\mathsf{a}_i}{\longrightarrow} & \mathsf{Q}_i & \stackrel{\mathsf{b}_i}{\longleftarrow} & \mathsf{U}_{i+1} & \stackrel{\mathsf{a}_{i+1}}{\longrightarrow} & \mathsf{Q}_{i+1} \\ & & \downarrow & & & & \downarrow & & \\ \to & & \mathsf{F}_i & & \to & & \mathsf{F}_{i+1} & & \to \end{array}$$

We call this sequence the *well module* of $f_{\mathbb{A}}$, denoted as U. We remark that U is a special case of a zigzag module as introduced in [2]. It is special because all forward maps are surjective and all backward maps are injective. Equivalently, there are no births other than at U_0.

Left filtration. Perhaps surprisingly, the evolution of the homology classes can still be fully described by pairing births with deaths, just like for a filtration. To shed light on this construction, we follow [2] and turn a zigzag module into a filtration. In our case, all births happen at U_0, so this transformation is easier than for general zigzag modules. Write $\mathsf{u}_{0,i} : \mathsf{U}_0 \to \mathsf{F}_i$ for the restriction of $\mathsf{f}_{0,i}$ to the initial well group. By the Shrinking Wellness Lemma, the image of this map contains the i-th well group, that is, $\mathsf{U}_i \subseteq \mathsf{u}_{0,i}(\mathsf{U}_0)$. We consider the preimages of the well groups in U_0 together with the preimages of their vanishing subgroups, $\mathsf{A}_i = \mathsf{u}_{0,i}^{-1}(\mathsf{K}_i)$ and $\mathsf{B}_i = \mathsf{u}_{0,i}^{-1}(\mathsf{U}_i)$; see Figure 4. We note that $\mathsf{A}_i / \mathsf{A}_{i-1} \simeq \ker \mathsf{a}_i$ and $\mathsf{B}_i / \mathsf{B}_{i+1} \simeq \operatorname{cok} \mathsf{b}_i$. In words, the first quotient represents the homology classes that die a conventional death and the second quotient represents the homology classes that die an unconventional death. As illustrated in Figure 4, the preimages form a nested sequence

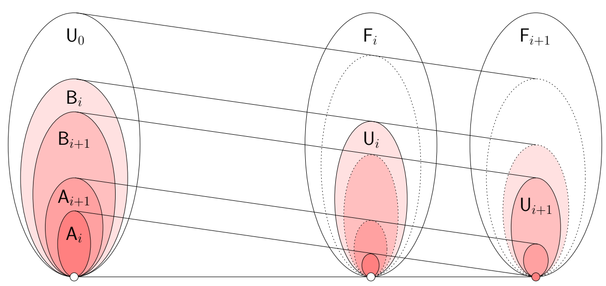

Figure 4: The left filtration decomposes U_0 into the preimages of the well groups and the preimages of their vanishing subgroups.

of subgroups of U_0. Together with the inclusion maps, this gives the *left filtration* of the zigzag module,

$$0 \to \mathsf{A}_0 \to \ldots \to \mathsf{A}_{l+1} = \mathsf{B}_{l+1} \to \ldots \to \mathsf{B}_0 = \mathsf{U}_0.$$

We can recover the well groups with $\mathsf{U}_i \simeq \mathsf{B}_i/\mathsf{A}_{i-1}$. Recall that $\mathsf{U}_{l+2} = 0$, which implies $\mathsf{K}_{l+1} = \mathsf{U}_{l+1}$. It follows that the middle two groups in the left filtration, A_{l+1} and B_{l+1}, are indeed equal.

Compatible bases. A useful property of the left filtration is the existence of compatible bases of all its groups. By this we mean a basis of U_0 that contains a basis for each A_i and each B_i. Specifically, we rewrite U_0 as a direct sum of kernels of forward maps and cokernels of backward maps:

$$\mathsf{U}_0 \quad \simeq \quad \ker \mathsf{a}_0 \oplus \ldots \oplus \ker \mathsf{a}_{l+1} \oplus \operatorname{cok} \mathsf{b}_l \oplus \ldots \oplus \operatorname{cok} \mathsf{b}_0.$$

Reading this decomposition from left to right, we encounter the A_i and the B_i in the sequence they occur in the left filtration. Choosing a basis for each kernel and each cokernel, we thus get compatible bases for all groups in the left filtration. We call this the *left filtration basis* of U_0. It is unique up to choosing bases for the kernels and cokernels.

Consider now a homology class α in U_0 and its representation as a sum of basis vectors. We write $\alpha(\mathsf{a}_i)$ for the projection of α to the kernel of the i-th forward map, which is obtained by removing all vectors that do not belong the the basis of $\ker \mathsf{a}_i$. Similarly, we write $\alpha(\mathsf{b}_i)$ for the projection of α to $\operatorname{cok} \mathsf{b}_i$. Letting j be the minimum index such that $\alpha(\mathsf{a}_i) = \alpha(\mathsf{b}_i) = 0$ for all $i \geq j$, we say that α *falls ill* at u_j.

Well diagrams. Constructing the birth-death pairs that describe the well module is now easy. All classes are born at U_0, however, to distinguish the changes in the well group from those in the homology group, we say all the classes *get well* at U_0. They fall ill later, and once they fall ill, they do not get well any more. The drop in rank from U_i to U_{i+1} is $\mu_i = \operatorname{rank}(\ker \mathsf{a}_i) + \operatorname{rank}(\operatorname{cok} \mathsf{b}_i)$. We thus have μ_i copies of the point $(0, u_i)$ is the diagram. There is no information in the first coordinates, which are all zero. We thus define the *well diagram* as the multiset of points u_i with multiplicities μ_i, denoting it as $\operatorname{Dgm}(\mathsf{U})$. For technical reasons that will become obvious in the next section, we add infinitely many copies of 0 to this diagram. Hence, each point in $\operatorname{Dgm}(\mathsf{U})$ is either 0, a positive real number, or ∞, and the diagram itself is a multiset of points on the extended line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. It has infinitely many points at 0 and a finite number of non-zero points.

As suggested by the heading of this section, we think of each point in the diagram as a measure for how resistant a homology class of $f^{-1}(\mathbb{A})$ is against perturbations of the mapping. At each well group U_i, an entire set of homology classes falls ill, and we call u_i the *robustness* of each class α in this set, denoting it as $\varrho(\alpha) = u_i$.

4 Proving Stability

We are interested in relating the difference between mappings to the difference between their well diagrams. After quantifying these differences, we connect parallel well modules to form new modules, and we finally prove that the well diagram is stable.

Distance between functions. Let $\mathbb{X}$ be an m-manifold, $\mathbb{Y}$ a Riemannian n-manifold, and $\mathbb{A} \subseteq \mathbb{Y}$ a k-manifold. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be two admissible mappings and assume they are homotopic. Recall that the distance between f and g is quantified by taking the largest distance between corresponding images in $\mathbb{Y}$, that is,

$$\|f - g\|_\infty \quad = \quad \sup_{x \in \mathbb{X}} \|f(x) - g(x)\|_\mathbb{Y}.$$

Using $\mathbb{A}$, we get two functions, $f_\mathbb{A}, g_\mathbb{A} : \mathbb{X} \to \mathbb{R}$. Similar to the mappings, the distance between them is the largest difference between corresponding values, that is,

$$\|f_\mathbb{A} - g_\mathbb{A}\|_\infty \quad = \quad \sup_{x \in \mathbb{X}} |f_\mathbb{A}(x) - g_\mathbb{A}(x)|.$$

The two distances are related. Specifically, the distance between the functions cannot exceed the distance between the mappings.

DISTANCE LEMMA. Let $f_\mathbb{A}, g_\mathbb{A} : \mathbb{X} \to \mathbb{R}$ be the functions defined by the mappings $f, g : \mathbb{X} \to \mathbb{Y}$ and the submanifold $\mathbb{A} \subseteq \mathbb{Y}$. Then $\|f_\mathbb{A} - g_\mathbb{A}\|_\infty \leq \|f - g\|_\infty$.

PROOF. We prove a stronger result, namely that the claimed inequality holds everywhere, that is,

$$|f_\mathbb{A}(x) - g_\mathbb{A}(x)| \quad \leq \quad \|f(x) - g(x)\|_\mathbb{Y} \tag{1}$$

at every point $x \in \mathbb{X}$. We may simplify this inequality by assuming that $f_\mathbb{A}(x) - g_\mathbb{A}(x)$ is non-negative. Suppose there exists a point $a \in \mathbb{A}$ for which $g_\mathbb{A}(x) = \|a - g(x)\|_\mathbb{Y}$. Being a metric, the distance in $\mathbb{Y}$ obeys the triangle inequality, and in particular

$$\|a - g(x)\|_\mathbb{Y} + \|g(x) - f(x)\|_\mathbb{Y} \quad \geq \quad \|a - f(x)\|_\mathbb{Y}.$$

The right hand side is an upper bound on $f_\mathbb{A}(x)$ which implies (1). Since we did not assume that $\mathbb{A}$ is compact, there might not be a point at which $g(x)$ attains its distance to $\mathbb{A}$. But for every $\delta > 0$, there is a point $a \in \mathbb{A}$ such that $g_\mathbb{A}(x) + \delta \geq \|a - g(x)\|_\mathbb{Y}$. Plugging this into the triangle inequality above gives $f_\mathbb{A}(x) - g_\mathbb{A}(x) - \delta \leq \|f(x) - g(x)\|_\mathbb{Y}$. Letting δ go to zero, we get (1). $\square$

Distance between diagrams. Let $\mathsf{G}(r)$ be the homology group and $\mathsf{V}(r) \subseteq \mathsf{G}(r)$ the well group of $g_{\mathbb{A}}^{-1}[0,r]$. As for f, we insert quotients between contiguous well groups and connect them with forward and backward maps to form a well module, denoted as V. The corresponding well diagram, $\mathrm{Dgm}(\mathsf{V})$, is again a multiset of points in $\bar{\mathbb{R}}$, consisting of infinitely many copies of 0 and finitely many non-zero points. Recall that the bottleneck distance between the diagrams of f and g is the length of the longest edge in the minimizing matching. Because our diagrams are one-dimensional, the bottleneck distance is easy to compute. To describe the algorithm, we order the positive points in both diagrams, getting

$$\begin{array}{ccccccccc} 0 & \leq & u_1 & \leq & u_2 & \leq & \ldots & \leq & u_M; \\ 0 & \leq & v_1 & \leq & v_2 & \leq & \ldots & \leq & v_M, \end{array}$$

where we add zeros to make sure we have two sequences of the same length. The *inversion-free matching* pairs u_i with v_i for all i. We prove that this matching gives the bottleneck distance.

MATCHING LEMMA. Assuming the above notation, the bottleneck distance between $\mathrm{Dgm}(\mathsf{U})$ and $\mathrm{Dgm}(\mathsf{V})$ is equal to $\max_{1 \leq i \leq M} |u_i - v_i|$.

PROOF. For a given matching, we consider the vector of absolute differences, which we sort largest first. Comparing two such vectors lexicographically, we now prove that the inversion-free matching gives the minimum vector. This implies the claimed inequality,

$$W_\infty(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})) = \max_{1 \leq i \leq M} |u_i - v_i|,$$

To prove minimality, we consider a matching that has at least one inversion, that is, pairs (u_i, v_t) and (u_j, v_s) with $i < j$ and $s < t$. If $u_i = u_j$ or $v_s = v_t$ then switching to the pairs (u_i, v_s) and (u_j, v_t) preserves the sorted vector of absolute differences. Otherwise, the new vector is lexicographically smaller than the old vector. Indeed, the minimum of the four points is u_i or v_s and the maximum is u_j or v_t. If the minimum and the maximum are from opposite diagrams then they delimit the largest of the four absolute differences, and this largest difference belongs to the old vector. Otherwise, both absolute differences shrink when we switch the pairs. Repeatedly removing inversions as described eventually leads to the inversion-free matching, which shows that it minimizes the vector and its largest entry is the bottleneck distance. ◻

Bridges. The main tool in the proof of stability are short bridges between parallel filtrations. The length of these bridges relates to the distance between the functions defining the filtrations. Let $\varepsilon = \|f - g\|_\infty$. By the Distance Lemma, we have $\|f_{\mathbb{A}} - g_{\mathbb{A}}\|_\infty \leq \varepsilon$, which implies that the sublevel set of $g_{\mathbb{A}}$ for radius r is contained in the sublevel set of $f_{\mathbb{A}}$ for radius $r + \varepsilon$. Hence, there is a homomorphism $\mathcal{B}_r : \mathsf{G}(r) \to \mathsf{F}(r+\varepsilon)$, which we call the *bridge* from G to F at radius r. We use the bridge to connect the initial segment of G to the terminal segment of F. The endpoints of the bridge satisfy the property expressed in the Shrinking Wellness Lemma.

BRIDGE LEMMA. Let $\mathcal{B}_r : \mathsf{G}(r) \to \mathsf{F}(r+\varepsilon)$ be the bridge at r, where $\varepsilon = \|f - g\|_\infty$. Then $\mathsf{U}(r+\varepsilon) \subseteq \mathcal{B}_r(\mathsf{V}(r))$.

PROOF. Let α be a homology group in $\mathsf{U}(r+\varepsilon)$. By definition of well group, there is a sufficiently small $\delta > 0$ such that α belongs to the image of $\mathsf{H}(h^{-1}(\mathbb{A}))$ in $\mathsf{F}(r+\varepsilon)$ for every $(r+\varepsilon+\delta)$-perturbation h of f. This includes all $(r+\delta)$-perturbations of g. It follows that the preimage of α in $\mathsf{G}(r)$ belongs to the well group, that is, $\mathcal{B}_r^{-1}(\alpha) \in \mathsf{V}(r)$. ◻

Everything we said about bridges is of course symmetric in F and G. In other words, $f_{\mathbb{A}}^{-1}[0,r] \subseteq g_{\mathbb{A}}^{-1}[0, r+\varepsilon]$ and there is a bridge from $\mathsf{F}(r)$ to $\mathsf{G}(r+\varepsilon)$ for every $r \geq 0$.

New modules. We use the Bridge Lemma to construct new zigzag modules from the well modules of f and g. Specifically, we use $\mathcal{B}_r$ to connect the initial segment of V, from $\mathsf{V}(0)$ to $\mathsf{V}(r)$, to the terminal segment of U, from $\mathsf{U}(r+\varepsilon)$ to $\mathsf{U}(\infty)$. To complete the module, we insert $\mathsf{Q}(r) = \mathsf{V}(r)/\ker \mathcal{B}_r$ between $\mathsf{V}(r)$ and $\mathsf{U}(r+\varepsilon)$. The forward map, from $\mathsf{V}(r)$ to $\mathsf{Q}(r)$, is surjective, and the backward map, from $\mathsf{U}(r+\varepsilon)$ to $\mathsf{Q}(r)$ is injective. The new zigzag module is thus of the same type as the well modules implying it has a left filtration basis that gives rise to a family of compatible bases for the groups in the left filtration.

A particular construction starts with the filtrations $\mathsf{F}(0) \to \ldots \to \mathsf{F}(\infty)$ and $\mathsf{G}(0) \to \ldots \to \mathsf{G}(\infty)$ and adds $\mathcal{B}_0 : \mathsf{G}(0) \to \mathsf{F}(\varepsilon)$. Following the bridge from G to F at 0, we get a new filtration and a new zigzag module, denoting the latter as W; see Figure 5. The decomposition of $\mathsf{W}(0) = \mathsf{V}(0)$ by

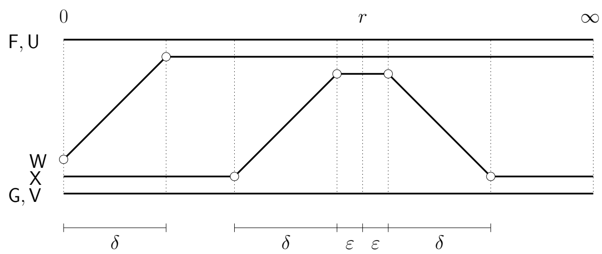

Figure 5: The four curves represent four filtrations as well as four the zigzag modules. The middle two are constructed from the outer two by adding bridges connecting the dots.

the left filtration of W is similar to the decomposition of $\mathsf{U}(0)$ by the left filtration of U; see Figure 4. Letting i be the index such that $u_i \leq \varepsilon < u_{i+1}$, we have $\mathsf{F}(\varepsilon) = \mathsf{F}_i$ and $\mathsf{U}(\varepsilon) = \mathsf{U}_i$. The classes in A_{i-1} and in $\mathsf{U}_0/\mathsf{B}_i$ die before we reach $\mathsf{F}(\varepsilon)$. The remaining classes form $\mathsf{U}(\varepsilon) \simeq \mathsf{B}_i/\mathsf{A}_{i-1}$. Correspondingly, there are homology classes in $\mathsf{W}(0)$ that die before we reach $\mathsf{F}(\varepsilon)$, namely the ones in the kernel of the forward map, from $\mathsf{W}(0)$ to $\mathsf{Q}(0)$, and in the preimage of the cokernel of the backward map, from $\mathsf{U}(\varepsilon)$ to $\mathsf{Q}(0)$. The remaining classes form $\mathsf{W}(\varepsilon) \simeq \mathcal{B}_0^{-1}(\mathsf{U}(\varepsilon))/\ker \mathcal{B}_0$. The two quotient groups, $\mathsf{U}(\varepsilon)$ and $\mathsf{W}(\varepsilon)$, are decomposed in parallel so that choosing a basis for $\mathsf{U}(\varepsilon)$ gives one for $\mathsf{W}(\varepsilon)$. This will be useful shortly.

Main result. We are now ready to state and prove the stability of the well diagram.

STABILITY THEOREM FOR WELL DIAGRAMS. Let U, V be the well modules of the functions $f_{\mathbb{A}}, g_{\mathbb{A}}$ defined by the admissible, homotopic mappings $f, g : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{A} \subseteq \mathbb{Y}$ are manifolds of finite dimension and $\mathbb{Y}$ is Riemannian. Then $W_\infty(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})) \leq \|f - g\|_\infty$.

PROOF. We construct a bijection from $\mathrm{Dgm}(\mathsf{U})$ to $\mathrm{Dgm}(\mathsf{V})$ such that the L_∞-distance between matched points is at most $\varepsilon = \|f - g\|_\infty$. Specifically, we match each point $u \leq \varepsilon$ in $\mathrm{Dgm}(\mathsf{U})$ with a copy of 0 in $\mathrm{Dgm}(\mathsf{V})$, and we use the parallel bases of $\mathsf{U}(\varepsilon)$ and $\mathsf{W}(\varepsilon)$ for the rest, where W is the zigzag module obtained by adding the bridge from G to F at radius 0, as described above.

Let α belong to the left filtration basis of $\mathsf{U}(0)$ such that its image belongs to the basis of $\mathsf{U}(\varepsilon)$. Let r be the value at which α falls ill and note that $r > \varepsilon$. Let β belong to the left filtration basis of $\mathsf{V}(0) = \mathsf{W}(0)$ such that the images of α and β in $\mathsf{W}(\varepsilon) = \mathsf{U}(\varepsilon)$ coincide. We now construct yet another zigzag module, by adding a first bridge from $\mathsf{G}(r - \varepsilon - \delta)$ to $\mathsf{F}(r - \delta)$ and a second bridge from $\mathsf{F}(r + \delta)$ back to $\mathsf{G}(r + \varepsilon + \delta)$, where $\delta > 0$ is sufficiently small such that there are no deaths in the interval $[r - \delta, r + \delta]$, except possibly at r. We denote the resulting module by X; see Figure 5. We note that all maps between groups are induced by inclusion so that the diagram formed by the filtrations and the bridges between them commutes.

By construction, the image of β in $\mathsf{F}(r - \delta)$ is non-zero and belongs to $\mathsf{U}(r - \delta)$. In contrast, the image of β in $\mathsf{F}(r + \delta)$ is either zero or lies outside $\mathsf{U}(r + \delta)$. Applying the Bridge Lemma going backward along the first bridge, we note that the image of $\beta \in \mathsf{W}(0) = \mathsf{X}(0)$ in $\mathsf{G}(r - \varepsilon - \delta)$ is non-zero and belongs to $\mathsf{V}(r - \varepsilon - \delta)$. Applying the Bridge Lemma going forward along the second bridge, we note that the image of β in $\mathsf{G}(r + \varepsilon + \delta)$ is either zero or lies outside $\mathsf{V}(r + \varepsilon + \delta)$. Since we can choose $\delta > 0$ as small as we like, this implies that β falls ill somewhere in the interval $[r - \varepsilon, r + \varepsilon]$. In the matching, this radius is paired with r, the radius at which α falls ill in U. The absolute difference between the two radii is at most ε, as required. ◻

5 Applications

In this section, we use the stability of the transversality measure to derive stability results for fixed points, periodic orbits, and apparent contours. All three problems can be recast in terms of intersections between manifolds and are therefore amenable to the tools developed in this paper.

Fixed points. A *fixed point* of a continuous mapping from a topological space to itself is a point that is its own image. Assuming this space is the m-dimensional Euclidean space and b is the mapping, we introduce a mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ defined by $f(x) = x - b(x)$. A fixed point of b is a root of f, that is, $f(x) = 0$. Writing $\mathbb{X} = \mathbb{Y} = \mathbb{R}^m$ and $\mathbb{A} = \{0\}$, the origin of $\mathbb{R}^m$, we get the setting studied in this paper. Each fixed point x of b corresponds to a class in the 0-dimensional homology group of $f^{-1}(0)$. Using the methods of this paper, we assign a non-negative robustness measure, $\varrho(x)$, to x. It gives the magnitude of perturbation necessary to remove this fixed point. This does not mean that a perturbation of smaller magnitude has a fixed point at precisely the same location but rather that it has one or more fixed points in lieu of x. Letting $\varrho(x)$ be the maximum robustness of all fixed points, then this implies that every $\varrho(x)$-perturbation of f has at least one fixed point. This implication suffices to give a new proof of a classic topological result on fixed points. Let $\mathbb{B}^m$ be the closed unit ball in $\mathbb{R}^m$.

BROUWER'S FIXED POINT THEOREM. Every continuous mapping $b : \mathbb{B}^m \to \mathbb{B}^m$ has a fixed point.

PROOF. Extend b to a mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$ by defining $b(x)$ equal to its value at $x/\|x\|_2$ whenever $x \not\in \mathbb{B}^m$. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $f(x) = x - b(x)$ and let $g : \mathbb{R}^m \to \mathbb{R}^m$ be the identity, defined by $g(x) = x$. We may assume that f is admissible, else the homology group of $f^{-1}(0)$ has infinite rank and f has infinitely many roots. The other mapping, g, is clearly admissible, with a single root at $x = 0$. The distance between the two mappings is

$$\begin{aligned}\|f - g\|_\infty &= \sup_{x \in \mathbb{R}^m} \|f(x) - g(x)\|_2 \\ &= \sup_{x \in \mathbb{R}^m} \|b(x)\|_2,\end{aligned}$$

which is at most 1. The well diagram of the identity consists of a single, non-zero point at plus infinity. The Stability Theorem for Well Diagrams implies that the well diagram of f

also has a point at plus infinity. But this implies that f has a root and, equivalently, that b has a fixed point. ◻

The above reduction of fixed points to a transversality setting uses the difference between two points, an operation not available if the mapping $b : \mathbb{M} \to \mathbb{M}$ is defined on a general Riemannian manifold. In this case, we can use the correspondence between the fixed points of b and the intersection points between the graph of b and the diagonal in $\mathbb{M} \times \mathbb{M}$. To apply the results of this paper, we set $\mathbb{X} = \mathbb{M}$, $\mathbb{Y} = \mathbb{M} \times \mathbb{M}$, and $\mathbb{A} = \{(x', x') \mid x' \in \mathbb{M}\}$. Furthermore, we define the distance between two points $x = (x', x'')$ and $y = (y', y'')$ in $\mathbb{M} \times \mathbb{M}$ equal to

$$\|x - y\|_{\mathbb{Y}} = \begin{cases} \infty & \text{if } x' \neq y'; \\ \|x'' - y''\|_{\mathbb{M}} & \text{if } x' = y'. \end{cases}$$

It is not difficult to see that this setting gives the same robustness values for the case $\mathbb{M} = \mathbb{R}^m$ discussed above.

Periodic orbits. We generalize the above setting by allowing for fixed points of iterations of the mapping. Letting $\mathbb{M}$ be a Riemannian manifold and $f : \mathbb{M} \to \mathbb{M}$ a mapping, we write $f^j : \mathbb{M} \to \mathbb{M}$ for the j-fold composition of f with itself. A sequence

$$F_j(x) = (x, f(x), f^2(x), \ldots, f^{j-1}(x))$$

is an *order-j periodic orbit* of f if $f^j(x) = f \circ f^{j-1}(x) = x$. It is straightforward to see the following relationship between f and its j-fold composite.

ORBIT LEMMA. A point $x \in \mathbb{M}$ is a fixed point of f^j iff $F_j(x)$ is an order-j periodic orbit of f.

We can therefore use the methods of this paper to measure the robustness of x, that is, to determine how much f^j needs to be perturbed to remove the fixed point. However, it would be more interesting to measure how much f needs to be perturbed to remove the periodic orbit. This is different because not every mapping can be written as the j-fold composite of another mapping. Adapting the framework accordingly is not difficult. Substituting perturbations h of f for those of f^j, we intersect the images of the homomorphisms induced by h^j. Call the resulting values the robustness of the periodic orbits of order j.

Apparent contours. As mentioned in the introduction, [8] reduces the stability of the contour of a mapping to the stability of well diagrams, the main result of this paper. We briefly review the reduction. Let $\mathbb{M}$ be a compact, orientable 2-manifold and $f : \mathbb{M} \to \mathbb{R}^2$ a smooth mapping. The derivative of f at a point x is a linear map from the tangent space to $\mathbb{R}^2$. The point x is *critical* if the derivative at x is not surjective, and the *apparent contour* of f is the set of images of critical points. Beyond smoothness of f, we assume that the distance functions it defines are admissible. Specifically, for each $a \in \mathbb{R}^2$, the function $f_a : \mathbb{M} \to \mathbb{R}$ is defined by mapping every point x to $f_a(x) = \|f(x) - a\|_2$ and we assume that $f_a^{-1}(0)$ consists of a finite number of points.

To study the apparent contour, we consider the entire 2-parameter family of distance functions. Fixing a value $a \in \mathbb{R}^2$, the sublevel sets of f_a form a filtration of homology groups and a zigzag module of well groups. Each point in the preimage of a falls ill at a particular radius interpreted as the robustness of that point. The main result of this paper implies that this measure is stable, that is, $W_\infty(\mathrm{Dgm}(\mathsf{U}), \mathrm{Dgm}(\mathsf{V})) \leq \|f_a - g_a\|_\infty$, where U and V are the well modules defined by the mappings $f, g : \mathbb{M} \to \mathbb{R}$ and the value $a \in \mathbb{R}^2$. As shown in [8], this implies that the apparent contours of f and of g are close. The sense in which they are close is interesting in its own right and we refer to that paper for details.

6 Discussion

The main contribution of this paper is the definition of a robustness measure for the homology of the intersection between manifolds, and a proof that this measure is stable. The question arises how different robustness is from persistence and whether there is a reduction of one to the other. We describe a setting in which the two are almost the same. Let $\mathbb{X}$ be a manifold, $\mathbb{Y} = \mathbb{R}$, and $\mathbb{A} = (-\infty, a]$. In the persistence diagram of $f : \mathbb{X} \to \mathbb{R}$, the points in $[-\infty, a] \times (a, \infty]$ correspond to classes alive at a. In other words, the quadrant represents the homology group $\mathsf{F}(0) = \mathsf{H}(\mathbb{A})$. Assuming a is not a terminal critical value of $f_{\mathbb{A}}$, this is also the initial well group, $\mathsf{U}(0) = \mathsf{F}(0)$. A point (r_b, r_d) in this quadrant satisfies $r_b \leq a < r_d$, and a class α represented by this point falls ill at $\varrho(\alpha) = \min\{r_d - a, a - r_b\}$. In other words, the robustness of α can be computed from its birth and death values in the filtration of sublevel sets. We know of no such reduction to persistence in more general settings. Perhaps, robustness sits somewhere between the classic 1-parameter notion of persistence and the algebraically much less tractable multi-parameter generalization [3]. Besides staking out this landscape, the results in this paper raise a number of questions and invite extensions in several directions.

- There are no principle obstacles to generalizing the notion of robustness to non-manifold spaces. Are there applications that can drive this extension or is it feasible to ask for a most general setting in which our framework is meaningful?
- Fixed points of mappings play an important role in game theory [16]. Can the results of this paper be used

to gain better insights into the nature of fixed points as they arise in different games? What are contexts in which the robustness of a fixed point is relevant to the understanding of the dynamics of a game?

- The three applications sketched in Section 5 barely scratch the surface of the possible. An interesting direction for further research are mappings from lower to higher dimensions. For example, the boundary of a computer-aided design model is the image of a mapping from a 2-manifold to $\mathbb{R}^3$. Can our results be used to detect and remove accidental self-intersections, a problem of significant economic importance [10].
- Except for a few special settings, we have no algorithms for computing well diagrams. The main obstacle is the infinite set of perturbations that appears in the definition of well groups. However, since the groups that arise for admissible mappings are finite, only a finite number of perturbations are relevant. Can we approach the algorithmic question from this direction?

References

[1] V. I. ARNOLD. *Catastrophe Theory.* Third edition, Springer-Verlag, Berlin, Germany, 1992.

[2] G. CARLSSON AND V. DE SILVA. Zigzag persistence. Manuscript, Dept. Mathematics, Stanford Univ., Stanford, California, 2008.

[3] G. CARLSSON AND A. ZOMORODIAN. The theory of multidimensional persistence. *In* "Proc. 23rd Ann. Sympos. Comput. Geom., 2007", 184–193.

[4] D. COHEN-STEINER, H. EDELSBRUNNER AND J. HARER. Stability of persistence diagrams. *Discrete Comput. Geom.* **37** (2007), 103–120.

[5] H. EDELSBRUNNER AND J. HARER. Persistent homology — a survey. *Surveys on Discrete and Computational Geometry. Twenty Years Later*, eds. J. E. Goodman, J. Pach and R. Pollack, Contemporary Mathematics **453**, 257–282, Amer. Math. Soc., Providence, Rhode Island, 2008.

[6] H. EDELSBRUNNER AND J. HARER. *Computational Topology. An Introduction.* Amer. Math. Soc., Providence, Rhode Island, 2009.

[7] H. EDELSBRUNNER, D. LETSCHER AND A. ZOMORODIAN. Topological persistence and simplification. *Discrete Comput. Geom.* **28** (2002), 511–533.

[8] H. EDELSBRUNNER, D. MOROZOV AND A. PATEL. The stability of the apparent contour of an orientable 2-manifold. Manuscript, Dept. Comput. Sci., Duke Univ., Durham, North Carolina, 2009.

[9] M. GOLUBITSKY AND V. GUILLEMIN. *Stable Mappings and Their Singularities.* Springer-Verlag, New York, 1973.

[10] H. GU, T. R. CHASE, D. C. CHENEY, T. BAILEY AND D. JOHNSON. Identifying, correcting, and avoiding errors in computer-aided design models which affect interoperability. *J. Comput. Inform. Sci. Engin.* **1** (2001), 156–166.

[11] V. GUILLEMIN AND A. POLLACK. *Differential Topology.* Prentice Hall, Englewood Cliffs, New Jersey, 1974.

[12] A. HATCHER. *Algebraic Topology.* Cambridge Univ. Press, Cambridge, England, 2002.

[13] K. POPPER. *The Logic of Scientific Discovery.* Basic Books, New York, New York, 1959.

[14] S. SMALE. Book review on *Catastrophe Theory: Selected Papers* by E. C. Zeeman. *Bull. Amer. Math. Soc.* **84** (1978), 1360–1468.

[15] R. THOM. *Structural Stability and Morphogenesis: an Outline of a General Theory of Models.* Translated from French by D. H. Fowler, Addison-Wesley, Reading, Massachusetts, 1989.

[16] A. VON SCHEMDE AND B. VON STENGEL. Strategic characterization of the index of an equilibrium. In "Sympos. Algor. Game Theory, 2008", 242–254, Lecture Notes Comput. Sci. **4997**, Springer-Verlag, Berlin, Germany.

[17] H. WHITNEY. The self-intersections of a smooth n-manifold in $2n$-space. *Annals of Math.* **45** (1944), 220–246,

[18] H. WHITNEY. On singularities of mappings of Euclidean space. I. Mappings of the plane to the plane. *Ann. Math.* **62** (1955), 374–410.

[19] E. C. ZEEMAN, ED. *Catastrophe Theory: Selected Papers, 1972-1977.* Addison-Wesley, Reading, Massachusetts, 1977. London, England, 1978.